DC Circuits

Cambridge Physics Academy

0 Readings

- Ch 31 of HRK
- Ch 4 of Purcell

Feel free to do problems from the readings as extra practice.

1 Lecture review

The most basic element in a circuit is the Resistor. Resistors are elements that follow Ohm's law.

Definition 1.1 (Ohm's Law)
In an Ohmic material, the current density and electric field are related by

$$\mathbf{J} = \sigma \mathbf{E},$$

where σ is the conductivity, equal to $1/\rho$, where ρ is the resistivity. At the macroscopic level, for an entire circuit element, this equation will lead to the more familiar Ohm's Law,

$$V = IR,$$

where R is the **resistance**, a value related to the resistivity and the dimensions of the circuit element.

Besides Ohm's law, the only other laws required to truly work with any type of circuit are Kirchoff's Laws. Anything else can be derived from these

Theorem 1.2 (Kirchoff's Laws)
They are as follows:

1. Junction Rule: the sum of currents into any node is 0. This is simply a restatement of the conservation of charge.
2. Loop Rule: the sum of potential differences around a loop is 0. This is a statement of conservation of energy or the conservative nature of the electric field (this idea will be more clear once we get to electromagnetic induction and inductors).

These two can derive the most basic circuit reductions,

Idea 1.3 (Series & Parallel)
The equivalent resistance of two resistors R_1 and R_2 in series is

$$R_{\text{eq}} = R_1 + R_2,$$

and in parallel it is

$$R_{\text{eq}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}.$$

Tip! (Circuit Tricks)
Now, although you can theoretically do any problem with these fundamental ideas, circuit problems can get quite tricky, so here are a couple tricks to remember.

1. If two points in a circuit have equal potential, you can freely attach and detach those two points.
2. If you have an infinite repeating resistor pattern, try recursion.
3. Sometimes you have to superpose different current injection arrangements, as we did in the infinite lattice problem.

Another method of dealing with circuits is using Thevenin and Norton equivalents. Though it is pretty unlikely that you will encounter this concept in olympiads, they still provide good intuition about circuits.

Theorem 1.4 (Thevenin & Norton Equivalents)
Any network composed of voltage sources, current sources, and resistors is equivalent to a single equivalent battery $\mathcal{E}_{\text{eq}}$ and a single equivalent resistor R_{eq}.

$$V(I) = \mathcal{E}_{\text{eq}} + IR_{\text{eq}}.$$

Any network is also equivalent to a single current source I_{eq} and a single equivalent resistor R_{eq}.

$$I(V) = I_{eq} + V/R_{eq}.$$

Some methods for finding the equivalents:

- Use Kirchhoff's law to reduce the circuit to the desired equation
- Faster tricks:
  - Consider an open circuit — the voltage across the output nodes is $\mathcal{E}_{\text{eq}}$ or $I_{\text{eq}}R_{\text{eq}}$.
  - Consider closing the circuit with a short — the current through is $\mathcal{E}_{\text{eq}}/R_{\text{eq}}$ or I_{eq}.

Definition 1.5 (Capacitors)
The capacitance of a conductor is defined as the ratio between its charge and potential,

$$C = \frac{Q}{V}.$$

For an object composed of two conductors, we define

$$C = \frac{Q}{\Delta V},$$

where Q is the magnitude of the charge on each and ΔV is the difference in potential.

Idea 1.6 (Energy in a Capacitor)
The energy stored in a capacitor is

$$U = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{Q^2}{2C}.$$

Idea 1.7 (RC Circuits)
If you connect a charged capacitor to a resistor in series, it will discharge exponentially,

$$Q(t) = Q_0 e^{-t/\tau}, \tau = RC,$$

where τ is called the time constant. The voltage and other components decay similarly. With more complicated resistor and capacitor setups, you should use Kirchoff's laws to write down differential equations and solve.

Idea 1.8 (Power)
The energy dissipated per time across any circuit element is equal to

$$P = IV.$$

This is because for a given charge q, the change in energy is qV, so the rate of change by taking the derivative is IV. For a *resistor*, this can be written in the following equivalent forms,

$$P = IV = V^2/R = I^2R.$$

2 Problem Set

2.1 Exercises

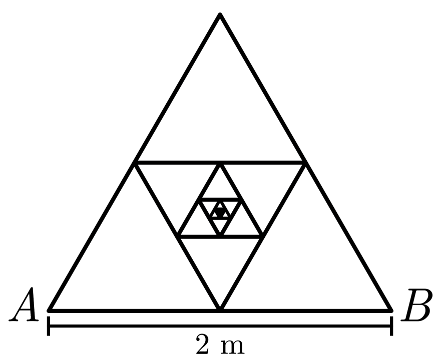
Problem 1. Consider a regular tetrahedron composed of 6 equal-length resistors, each of resistance R. What is the resistance between two adjacent vertices?

Solution. By symmetry, we can disconnect the opposing edge resistor. Thus, We have a parallel circuit with two branches of resistance $2R$ and one branch of resistance R. This computes to

$$R_{\text{eq}} = \frac{1}{1/R + 2/2R} = \boxed{R/2}.$$

■

Problem 2. Here's a classic resistor network problem which was given during camp to see how fast students finished. The circuit below is created by placing an infinite number of triangles of resistors, with each successive triangle being half the size of the previous. If the resistance per unit length of all segments in the circuit is ρ, find the resistance between points A and B.



Solution. First, using circuit trick #1, we can disconnect the bottom connection. Then, we can use recursion to figure out the resistance. Let the resistance between A and B be R_{eq}. Then, the resistance of the smaller inner triangle, whose edges resistances are all half of that of the full triangle, must be $R_{\text{eq}}/2$. Thus, we can replace the connection between the two upper midpoints with a resistor of value $R_{\text{eq}}/2$.

Now, we just write down R_{eq} in a second way by reducing the circuit further. This is just using series and parallel laws,

$$R_{\text{eq}} = \frac{1}{\frac{1}{2\rho} + \frac{1}{2\rho + \frac{1}{\frac{1}{2\rho} + \frac{1}{R_{\text{eq}}/2}}}}.$$

Ater some algebra, the above equation reduces to

$$R_{\text{eq}} = \frac{4\rho R_{\text{eq}} + 8\rho^2}{3R_{\text{eq}} + 8\rho}.$$

Multiplying this out, we get the quadratic

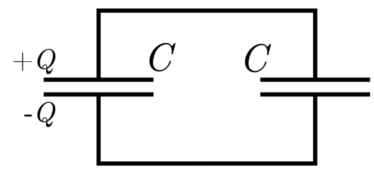
$$3R_{\text{eq}}^2 + 4\rho R_{\text{eq}} - 8\rho^2 = 0,$$

which has a positive solution of

$$R_{\text{eq}} = \boxed{\frac{-4 + 2\sqrt{7}}{3}\rho}.$$

■

Problem 3. Suppose you connect two capacitors of equal capacitance C in series with wires of negligible resistance, but you charge one of them up with charge Q before they are connected.



(a) After a long time, how much charge is on each capacitor?

(b) What is the change in total energy stored in the capacitors?

(c) Where does the energy loss come from? Can you show this quantitatively?

Solution.

(a) By symmetry and conservation of charge, since the voltage must be balanced on both sides, there will be $Q/2$ on each capacitor.

(b) Using the formula for energy, we have

$$\Delta U = 2 \times \frac{1}{2}\frac{(Q/2)^2}{C} - \frac{1}{2}\frac{Q^2}{C} = -\frac{1}{4}\frac{Q^2}{C}.$$

(c) The trick here is that if the resistance is really small the charge will shift really quickly with high current and so the energy dissipated through the wires will still be significant. Suppose that the wires have some small resistance r. We can write down the following loop rule,

$$\frac{Q_1}{C} + r\frac{dQ_1}{dt} - \frac{Q_2}{C} = 0.$$

We have by conservation of charge $Q_1 + Q_2 = Q$, so our diffeq is

$$\frac{2Q_1}{C} - \frac{Q}{C} + r\frac{dQ_1}{dt} = 0 \implies \frac{dQ_1}{dt} = \frac{Q - 2Q_1}{rC}.$$

Separating, we get

$$\frac{dQ_1}{2Q_1 - Q} = -\frac{dt}{rC},$$

which gives us

$$\frac{1}{2}\ln\left[\frac{2Q_1 - Q}{Q}\right] = -\frac{t}{rC} \implies Q_1 = \frac{Q}{2} + \frac{Q}{2}e^{-2t/rC}.$$

So, the current over time is

$$I(t) = -\frac{dQ_1}{dt} = \frac{Q}{rC}e^{-2t/rC}.$$

So, the power dissipated over time

$$P(t) = I^2 r = \frac{Q^2}{rC^2}e^{-4t/rC}.$$

Then, integrating that we get

$$-\Delta U = \int_0^\infty P(t)\,dt = \frac{Q^2}{rC^2}\frac{rC}{4} = \frac{Q^2}{4C},$$

matching the energy loss in part (b).

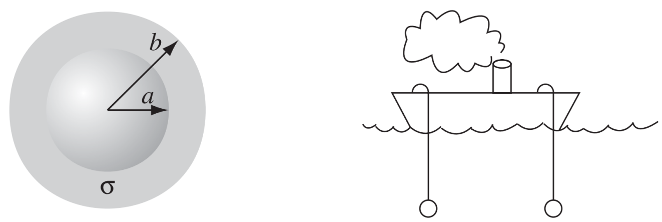
Problem 4. Consider the interface between a wire and a resistor. There is a change in the conductivity, $\sigma_w > \sigma_r$. By conservation of charge, the current density on both sides must be the same, which means that the electric field on both sides must be different. How can this be possible? What does the circuit look like before equilibrium?

Solution. the electric field being different means that there must be a plane of charge at the interface creating the change in electric field. Where does this plane of charge come from? Initially when the electric field is continuous through the interface, the current density in the wire will be greater than that of the resistor as

$$J_w = \sigma_w E > \sigma_r E = J_r.$$

So, there will be more charge coming into the interface than leaving, leading to the buildup of charge which creates teh change in the electric field. ■

Problem 5 (Griffiths)**.** Two concentric spherical conducting shells of radii a and b are separated by a material of resistivity ρ.



(a) (b)

(a) What is the resistance between the two shells?

(b) Now, suppose you have two metal spheres of radius a separated by a distance $d \gg a$ and submerged underwater (which has resistivity ρ) as shown above. What is the resistance between these two spheres? *Hint:* Consider the previous case with $b \gg a$. Alternatively, consider what the currents would look like with just one of the spheres at a time.

(c) (Unrelated) Now, if you removed the the material of resistivity ρ in part (a) and replaced it with air, what is the capacitance between the two spherical shells?

Solution.

(a) Assume that we run a current I through the inner sphere and take out I from the outer sphere. Then, the current density at a radius r is $J = I/(4\pi r^2)$. So, we can integrate the electric field to find the potential difference,

$$\Delta V = -\int_a^b \frac{I}{4\pi\sigma r^2}\, dr = \frac{I}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right),$$

so the resistance is

$$R = \frac{\Delta V}{I} = \frac{\sigma}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right).$$

(b) The trick here is to realize that in the case of $d \gg a$, you can consider the setup as two spherical resistors in series with $b \sim d \gg a$. This automatically gives you the answer

$$R_{\text{eq}} = \frac{2}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right) = \boxed{\frac{\rho}{2\pi a}}.$$

Otherwise, if you imagine I going in at one sphere then the current will spread out uniformly with

$$\mathbf{J} = \frac{I}{4\pi r^2}\hat{r}.$$

This is precisely the shape of the field for a point charge. So, if we take out I at the other sphere, the current lines will look just like that of two point charges. So, using the analogous formula, and noticing that at the surface of each sphere only the close point charge contributes to the potential significantly,

$$\Delta V = \frac{I\rho}{4\pi a} - \frac{-I\rho}{4\pi a} = \frac{I\rho}{2\pi a},$$

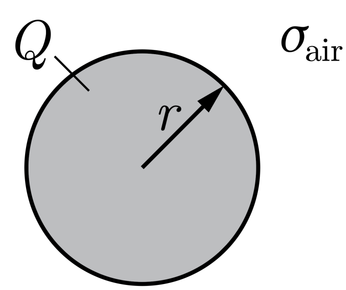
which gives us the same result.

(c) if there is a charge Q on the inner sphere, then the electric field as a function of r is $E = kQ/r^2$. Thus, the potential difference is

$$\Delta V = \int_a^b \frac{kQ}{r^2}\,dr = kQ\left(\frac{1}{a} - \frac{1}{b}\right).$$

Thus, the capacitance is $C = Q/V = \frac{4\pi\epsilon_0 ab}{b-a}$.

Problem 6 (MPPP / USAPhO)**.** Suppose that you have a conducting sphere of radius r which has a charge Q placed on it. It is surrounded by air which has conductivity σ_{air}.



(a) Working from first principle ($\mathbf{J} = \sigma\mathbf{E}$), find the time for the charge on the sphere to go down by a factor of e due to leakage to the air.

(b) Modelling the system as an RC circuit (imagine a conducting sphere of radius $R \approx \infty \gg r$), find the time constant, and show that it matches up with part (a).

(c) Now suppose that we have an arbitrarily-shaped conductor. Show that the time for the charge on the conductor to go down by a factor of e still remains the same. *Hint:* Some ideas from Electrostatics may be helpful.

Solution.

(a) At the boundary of the sphere we have that

$$\mathbf{E} = \frac{kQ}{r^2}\hat{r} \implies \mathbf{J} = \sigma_{\text{air}}\frac{kQ}{r^2}\hat{r}.$$

Thus, the rate of change of charge on the sphere is

$$\frac{dQ}{dt} = -J(4\pi r^2) = -\sigma_{\text{air}}(4\pi kQ).$$

This is a first order differential equation in Q, which we can solve by separating, and we get

$$Q(t) = Qe^{-4\pi k \sigma_{\text{air}} t}.$$

Thus, reading off of the equation, the time for the charge to go down by a factor of e is $\tau = \frac{1}{4\pi k \sigma_{\text{air}}} = \boxed{\frac{\epsilon_0}{\sigma_{\text{air}}}}$.

(b) Considering the hint given in the problem, we first have that the resistance is (using problem 5),

$$R = \frac{1}{4\pi\sigma_{\text{air}}}\left(\frac{1}{r} - \frac{1}{R}\right) \approx \frac{1}{4\pi\sigma_{\text{air}} r}.$$

Similarly, the capacitance is

$$C = \frac{r}{k}.$$

Thus, the time constant is $\tau = RC = \frac{1}{4\pi k\sigma_{\text{air}}} = \frac{\epsilon_0}{\sigma_{\text{air}}}$, matching part (a).

(c) We go back to part to the method of part (a). We have that the current density at the boundary of this surface is still

$$\mathbf{J} = \sigma \mathbf{E}.$$

So, the total current lost through the surface per time is given by

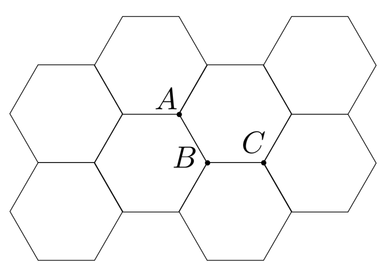
$$\frac{dQ}{dt} = -\oint_S \mathbf{J} \cdot d\mathbf{A} = -\oint_S \sigma \mathbf{E} \cdot d\mathbf{A} = -\sigma \oint \mathbf{E} \cdot \mathbf{A}.$$

But the integral is simply the flux, which we know from Gauss's law is Q/ϵ_0. Thus,

$$\frac{dQ}{dt} = -\sigma \frac{Q}{\epsilon_0},$$

which is exactly the same differential equation as in part (a), and thus the time constant is still $\tau = \epsilon_0/\sigma_{\text{air}}$.

Problem 7 (Kalda)**.** There is an infinite honeycomb lattice; the edges of the lattice are made of wire, and the resistance of each edge is R. Find the resistance between the two points labelled A and C.



Solution. We directly use the trick from class. Imagine injecting a current I into node A. Then, by symmetry there wll be current $I/3$ in $\overline{AB}$ and current $I/6$ in $\overline{BC}$. Thus, the potential difference between A and C in this case is

$$V_{AC}^{(\to A)} = \frac{I}{3}R + \frac{I}{6}R = \frac{IR}{2}.$$

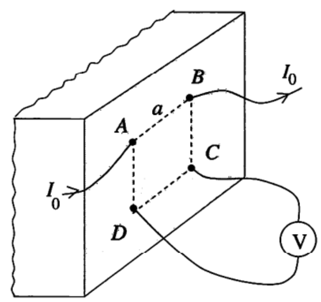
Similary, if we take out a current I from C, we would get that

$$V_{AC}^{(\to C)} = \frac{IR}{2}.$$

Thus, the total potential difference in the superposition case is $V_{AC} = IR/2 + IR/2 = IR$, and thus the equivalent resistance is $R_{\text{eq}} = \boxed{R}$. ■

Here's another problem on finding the resistance. All the ideas used for finding the resistances of other objects work, but this might require a couple other problem solving ideas we've covered in class.

Problem 8 (PPP)**.** A plane divides space into two halves. One half is filled with homogeneous conductin medium and physicists work in the other. They mark the outline of a square of side a on the plane an dlet a current I_0 in and out at two of its neighbouring corners using fine electrodes.



They measure the potential difference V_0 between the two other corners. What is the resistivity of the material in terms of I_0 and V_0?

Solution.

(a) Following the strategy of the previous problem, we consider teh case where we inject I_0 into A. By symmetry it spreads out evenly over any size hemisphere in the plane. Thus, the electric field is

$$E(r) = \rho J(r) = \rho \frac{I_0}{2\pi r^2}.$$

Thus, the potential difference in this case

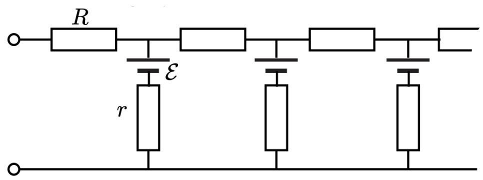
$$V_{DC}^{(A)} = \frac{\rho I_0}{2\pi} \left(\frac{1}{a} - \frac{\sqrt{2}}{a} \right)$$

So, the total potential difrence considering taking a curent I_0 out of B too becomes

$$V_0 = V_{DC} = \frac{\rho I_0}{\pi a}(1 - \sqrt{2}).$$

So, the resistivity in terms of I_0 and V_0 is $\rho = \frac{V_0}{I_0} \frac{\pi a}{1-\sqrt{2}}$.

Problem 9 (Kalda)**.** Find the thevenin equivalent EMF and resistance for the following infinite circuit below:



Solution. Suppose the thevenin equivalent EMF is V and the equivalent resistance is R_{eq}. Then, place these two in parallel with the first section, using recursion. Now, we use the normal thevenin equivalent strategies. If we measure the voltage in the closed circuit, because the current through the right loop from Ohm's law is

$$I = \frac{\mathcal{E} - V}{R_{\text{eq}} + r},$$

we have that the voltage is

$$V = \mathcal{E} - \frac{\mathcal{E} - V}{R_{\text{eq}} + r} r.$$

Thus,

$$(V - \mathcal{E}) = (V - \mathcal{E}) \frac{r}{R_{\text{eq}} + r},$$

which means that we must have $V - \mathcal{E} = 0 \implies \boxed{V = \mathcal{E}}$. Now, if we consider the short circuit case, we need to use Kirchoff's laws to figure out the current through the output nodes. Becuase the two batteries are now the same voltage, we can replace them with one and attach the wires, leaving the circuit as a battery with three resistors of equivalent resistance

$$R'_{\text{eq}} = R + \frac{1}{\frac{1}{r} + \frac{1}{R_{\text{eq}}}}.$$

However, since we have

$$\frac{V}{R_{\text{eq}}} = \frac{\mathcal{E}}{R'_{\text{eq}}} = \frac{V}{R'_{\text{eq}}},$$

we have $R'_{\text{eq}} = R_{\text{eq}}$, so

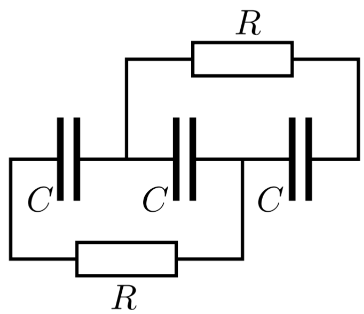
$$R_{\text{eq}} = R + \frac{1}{\frac{1}{r} + \frac{1}{R_{\text{eq}}}} \implies rR_{\text{eq}} + R_{\text{eq}}^2 = R(r + R_{\text{eq}}) + rR_{\text{eq}}.$$

Simplifying, and using the quadratic formula, we get

$$\boxed{R_{\text{eq}} = \frac{R + \sqrt{R^2 + 4Rr}}{2}}.$$

■

Problem 10 (Kalda). Three identical charge-less capacitors of capacitance C are connected in series. The capacitors are charged by connecting a battery of electromotive force $\mathcal{E}$ to the terminal leads of this circuit. Next, the battery is disconnected, and two resistors of resistance R are connected simultaneously as shown in figure below. Find the net heat which will be dissipated on each of the resistances.



Solution. When the capacitors are charged up, since they are simply in a line, each one just has voltage $\mathcal{E}/3$ with charge $C\mathcal{E}/3$. Assume that they have positive charge on their right plates.

Now, instead of actually finding the power dissipated on the resistors per time, we simply want to find what the final equilibrium state is and then find the difference in energy from the initial state. Suppose in the final state the charges on the three capacitors are q_1, q_2, q_3 with the value being positive if the right plate has positive charge. Then, first using the bottom loop, we get

$$\frac{q_1}{C} + \frac{q_2}{C} = 0 \implies q_1 = -q_2.$$

Using the top loop, we similarly get

$$\frac{q_2}{C} + \frac{q_3}{C} = 0 \implies q_3 = -q_2 = q_1.$$

Now, lastly, we use conservation of charge to get

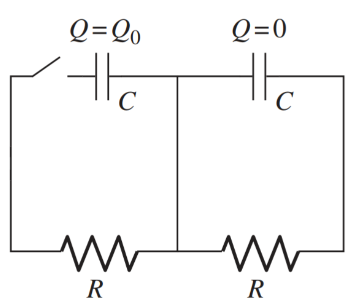
$$\begin{aligned}(-q_1) + q_2 + (-q_3) &= (-C\mathcal{E}/3) + (C\mathcal{E}/3) + (-C\mathcal{E}/3) \\ -3q_3 &= -C\mathcal{E}/3.\end{aligned}$$

So, $q_1 = q_3 = C\mathcal{E}/9$ and $q_2 = -C\mathcal{E}/9$. So, the change in energy is

$$\Delta U = \frac{1}{2C}\left(3 \times (C\mathcal{E}/3)^2 - 3 \times (C\mathcal{E}/9)^2\right) = \frac{1}{2C}\frac{8C^2\mathcal{E}^2}{27}.$$

And each resistor dissipates half of this, so $\boxed{Q = \frac{2}{27}C\mathcal{E}^2}$. ■

Problem 11 (Purcell)**.** The circuit below contains two identical capacitors and two identical resistors. Initially, the left capacitor has charge Q_0 (with the left plate positive), and the right capacitor is uncharged. If the switch is closed at $t = 0$, find the charges on the capacitors as functions of time. Your loop equations should be simple ones.



Solution. Let Q_1 denote the charge on the left capacitor and Q_2 denote the charge on the right capacitor. The loop equation on the left is

$$\frac{Q_1}{C} + R\frac{dQ_1}{dt} = 0,$$

and similarly on the right loop

$$\frac{Q_2}{C} + R\frac{dQ_2}{dt} = 0.$$

Thus, we see that the two halves of the circuit are entirely independent, so the left side just decays on its own (this makes sense as it would skip the right half of the circuit by just flowing current through the middle wire), so we have

$$Q_1(t) = Q_0 e^{-t/RC},$$
$$Q_2(t) = 0.$$

■

Problem 12. In this problem we explain why $\mathbf{J} \propto \mathbf{E}$ with some very rough estimations — do not worry about any numerical factors. Model a conductor as a bunch of particles of mass m, charge q, cross-sectional area σ, and number density n. Suppose that each time any two of these particles collide, each particle leaves in a randomized direction with a radomized speed, which is on average $\overline{v}$.

(a) Estimate the average distance λ that a particle travels before colliding with a different particle. We'll touch on this more in thermodynamics.

(b) Estimate the average time betwen collisions τ.

(c) Now, suppose that there is an external electric field $\mathbf{E}$ applied across the particles. Estimate the average *velocity* over all the particles. You'll have to consider the randomized portion due to collisions along with the directional portion due to the electric field.

(d) Write down the current density and give an estimate for ρ in terms of the parameters in the problem.

Solution.

(a) For simplicity, we focus on a single particle and assume that it moves through a still gel of particles with number density n. Then, we would like the average number of particles in the volume sweeped out by the particle as it travels a distance λ to be order 1. Thus, we write

$$\sigma\lambda n \sim 1 \implies \lambda \sim \frac{1}{\sigma n}.$$

(b) Dividing the average distance by the average speed, $\tau = \frac{\lambda}{\overline{v}} = \frac{1}{\sigma n \overline{v}}$.

(c) If we focus on a single particle, around every τ it will have a randomized velocity $\mathbf{u}_{\text{rand}}$. And through the τ as it doesn't collide, it will be accelerated by the electric field $\mathbf{E}$ for an anverage time τ. So, very roughly, the velocity of a single particle is

$$\mathbf{v} = \mathbf{u}_{\text{rand}} + \frac{q\mathbf{E}}{m}\tau.$$

So if we average over all particles,

$$\langle \mathbf{v} \rangle = \left\langle \mathbf{u}_{\text{rand}} + \frac{q\mathbf{E}}{m}\tau \right\rangle = \langle \mathbf{u}_{\text{rand}} \rangle + \left\langle \frac{q\mathbf{E}}{m}\tau \right\rangle.$$

The first term averages out to 0, and the second term is a constant, so we have that the average veloctiy of the particles in this set of particles is

$$\langle \mathbf{v} \rangle \sim \frac{q\mathbf{E}}{m}\tau.$$

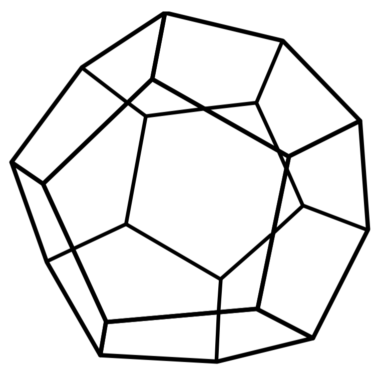
(d) We can write the current density as

$$\mathbf{J} = qn\mathbf{v} = \frac{q^2 n\tau}{m}\mathbf{E},$$

giving us our desired Ohm's law form! And the proportionality constant is $\rho = \frac{m}{q^2 n\tau} = \frac{\sigma m\bar{v}}{q^2}$.

2.2 Challenge Problems

Problem 13 (Kalda)**.** Determine the resistance between two neighbouring vertices of a dodecahedron (see figure), the edges of which are made of wire; the resistance of each edge is R.



This one's quite tricky! Make sure to ask for a hint if you need one.

Solution. We again want to use the current injection and superposition method. However, we can't simply inject current as this circuit is not infinite–there will just be charge buildup somewhere on the dodecahedron until the current can enter anymore, making it a non-static situation. So, we solve this by injecting a current I into one vertex and taking out a current $I/19$ from all the other vertices. This preserves the setup. In this case, we have

$$V_{AB}^{(}A) = \frac{IR}{3}, V_{AB}^{(}B) = \frac{IR}{3}.$$

For vertex B we take out a curent I and insert a current $I/19$ into all the other vertices. Now,, if we consider the superpositions, the current gonig in and out of all vertices is 0 except for A and B where it is $I + I/19 = 20I/19$. And we have

$$V_{AB} = 2 \times \frac{IR}{3},$$

so the resisitnace is $R = V/I = \frac{IR/3}{20I/19} = \frac{19R}{60}$.. ■

> **Remark:** Don't be scared by the "IPhO" label below! In fact, a lot of USAPhOs can be on par with the difficulty of IPhOs — IPhOs are generally just longer and combine a lot of concepts. The two problems below illustrate some really cool contraptions that arise from combining a lot of physics concepts — give them a good shot!

Problem 14. IPhO 2004, problem 1.

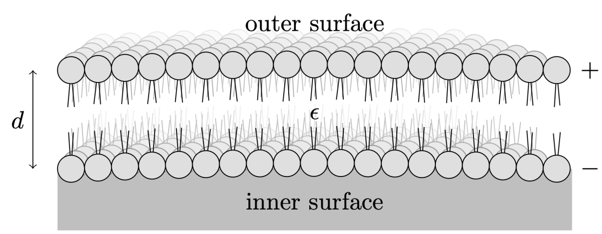
Problem 15. IPhO 1993, problem 1.

> **Remark:** Many of the trickiest problems in this problem set have come from the Kalda Circuits handout. If you found them really tough, don't worry too much as the USAPhO will most likely never have a problem of this style. But they are great brain teasers!

2.3 USAPhO Practice

Both are neuron-themed, but distinct!

Problem 16 (USAPhO 2019/B1)**.** The wall of a neuron is made from an elastic membrane, which resists compression in the same way as a spring. It has an effective spring constant k and an equilibrium thickness d_0. Assume that the membrane has a very large area A and negligible curvature. The neuron has "ion pumps" that can move ions across the membrane. In the resulting charged state, positive and negative ionic charge is arranged uniformly along the outer and inner surfaces of the membrane, respectively. The permittivity of the membrane is ϵ.



(a) Suppose that, after some amount of work is done by the ion pumps, the charges on the outer and inner surfaces are Q and $-Q$, respectively. What is the thickness d of the membrane?

(b) Derive an expression for the voltage difference V between the outer and inner surfaces of the membrane in terms of Q and the other parameters given.

(c) Suppose that the ion pumps are first turned on in the uncharged state, and the membrane is charged very slowly (quasistatically). The pumps will only turn off when the voltage difference across the membrane becomes larger than a particular value V_{th}. How large must the spring constant k be so that the ion pumps turn off before the membrane collapses?

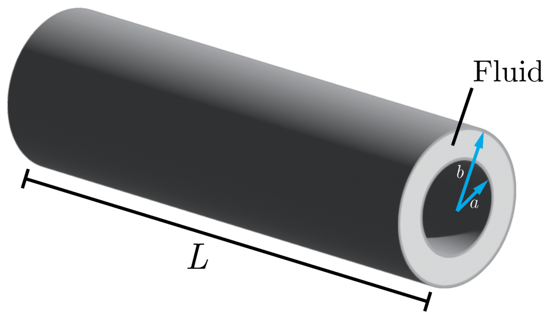
(d) How much work is done by the ion pumps in each of the following situations? Express your answers in terms of k and d_0.

   i. k is infinitesimally larger than the value derived in part (c).

   ii. k is infinitesimally smaller than the value derived in part (c).

Assume in each case that the membrane thickness d cannot become negative.

Problem 17. When a neuron is charged up to a threshold voltage, the voltage-gated channels open and the potential difference between the inner and outer membranes. All this occurs along the axon, which we'll model as a cylinder with inner radius a, outer radius b, and length $L \gg a, b$. The space between the two cylinders is filled with a fluid of dielectric constant κ.



(a) When the voltage-gated channels are opened, ions of mass m and charge q flow through the fluid in between. The fluid is viscous, so as they flow through, the ions experience a drag force of $F = -\eta v$. In terms of m, q, or η, what is the resistivity of the fluid?

(b) Using your result from part (a), what is the resistance R between the two cylinders?

(c) What is the capacitance C of the model if the inner and outer cylinders are taken as the two electrodes?

(d) Now, we model the neuron as an RC circuit.

i. Draw the effective circuit of the neuron, comprising of the resistor and capacitor found in the previous parts and potentially additional circuit components. Explain your circuit drawing briefly and why it corresponds to the model we've developed above. In particular, be sure to explain the rationale behind any wired connections in the circuit.

ii. Estimate the time constant of the circuit.

Solution.

(a) Once the fluid reaches a quasi-static equilibrium, we will have that the drag froce balances out the electric force. Thus, we'll have

$$\eta \mathbf{v} = q\mathbf{E}.$$

Now we want to use $\mathbf{v}$ to figure out the current density. A set of charges with velocity $\mathbf{v}$ and number density n passes an area A with rate

$$\frac{dN}{dt} = nAv.$$

Thus, the current through that section is $q\frac{dN}{dt}$, which let's us write

$$\mathbf{J} = \frac{I}{A} = \frac{qnA\mathbf{v}}{A} = qn\mathbf{v} = qn\frac{q\mathbf{E}}{\eta}.$$

Thus, we have the proportionality constant between $\mathbf{J}$ and $\mathbf{E}$ is q^2n/η, which is our conductivity. So, the resistivity is $\boxed{\rho = \eta/q^2n}$.

(b) If a current I was injected into the inner cylinder and taken out of the outside, the current density as a function of radius in the cylinder is

$$J(r) = \frac{I}{2\pi r L}.$$

Thus, the change in potential difference is

$$V = \int_a^b \frac{\rho I}{2\pi r L} = \frac{\rho I}{2\pi L}\ln(b/a),$$

so the resistance is

$$R = \frac{\rho}{2\pi L}\ln(b/a) = \frac{\eta}{2\pi q^2 n L}\ln(b/a).$$

(c) If we place a charge Q on the inner cylinder, then, the electric field as a function of radius, incorporating the dielectric constant is,

$$E(r) = \frac{1}{\kappa}\frac{Q/L}{2\pi\epsilon_0 r}.$$

Thus, the potential difference is

$$V = \int_a^b \frac{Q}{2\pi\epsilon_0 L r}\,dr = \frac{Q}{2\pi\kappa\epsilon_0 L}\ln(b/a).$$

So, the capacitance is

$$C = \frac{Q}{V} = \frac{2\pi\kappa\epsilon_0 L}{\ln(b/a)}.$$

(d) i. Here's the diagram of the circuit:

It's a bit confusing because, physically, the resistor is situated in between the two capacitor plates. However, if you actually think about the movement of charge, notice that the charge leaving the cpaacitor plates – the outside membranes – is equal to the amount that travels through the fluid – the resistor. Thus, we can treat the resistor as capacitor as in a loop.

ii. This is the simplest possible RC circuit, so the time constant is just

$$\tau = RC = \frac{\eta}{2\pi q^2 n L}\ln(b/a)\frac{2\pi\kappa\epsilon_0 L}{\ln(b/a)} = \frac{\kappa\epsilon_0\eta}{q^2 n}.$$

2.3.1 Optional, Additional Practice

Problem 18. USAPhO 2015, A2