Induction

Cambridge Physics Academy

0 Readings

- Ch 34-36 of HRK
- Ch 7 of Purcell

Feel free to do problems from the readings as extra practice.

1 Lecture review

Definition 1.1 (Electromotive Force)
The electromotive force, denoted by $\mathcal{E}$ is a generalization of the potential difference. Instead of just looking at the the electric field, it looks at the force per charge from the lorentz force:

$$\mathcal{E} = \int ((\mathbf{v} \times \mathbf{B}) + \mathbf{E}) \cdot d\mathbf{r}.$$

For a fixed loop moving in a temporally constant magnetic field (not necessarily spatially), integrating the expression above (check this as an exercise for yourself!) gives,

$$\mathcal{E} = \int (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r} = -\frac{d\Phi_B}{dt}.$$

More generally, for a non-temporally constant magnetic field, we have Faraday's law.

Theorem 1.2 (Faraday's Law)
In the case of a changing magnetic field and/or a loop moving in a non-uniform magnetic field, the emf induced in the loop is given by

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \iff \oint \mathbf{E} \cdot d\mathbf{r} = -\frac{d\Phi_B}{dt}.$$

The negative sign in Faraday's Law above indicates **Lenz's Law**: the emf induced in a loop of wire is such that the generated magnetic field opposes the change in the external magnetic flux.

Definition 1.3 (Self-Inductance)
A circuit element can *induce* a current within itself through its self-generated magnetic field. This is characterized by the inductance, which is defined as

$$\Phi_B = LI, V = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}.$$

The energy stored within an inductor is given by

$$U = \frac{1}{2}LI^2.$$

Definition 1.4 (Mutual Inductance)
If you have two inductors of inductance L_1, L_2, then the total magnetic flux through each is given by

$$\Phi_1 = L_1 I_1 + M_{12} I_2,$$
$$\Phi_2 = L_2 I_2 + M_{21} I_1.$$

Theorem 1.5
The mutual inductances are always equal,

$$M_{12} = M_{21}.$$

Idea 1.6 (LR Circuits)
If you connect a inductor with constant current to a resistor in series, its current will decay exponentially,

$$I(t) = I_0 e^{-t/\tau}, \tau = L/R,$$

where τ is the time constant. Intuitively, the inductor always wants to keep the same amount of current flowing through it.

2 Problem Set

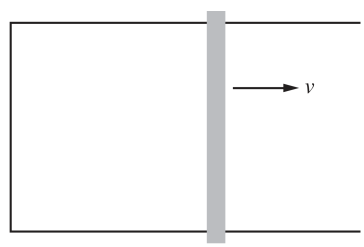
2.1 Exercises

Problem 1. In class we determined the effective inductance of two inductors where we assumed the mutual inductance was negligible. This time, assume that the mutual inductance between the two inductors with inductances L_1 and L_2 is a non-negligible value $M = M_{12} = M_{21}$, and find the total effective inductance when

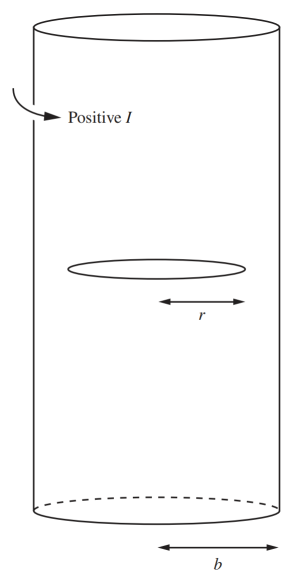
(a) They are connected in series.

(b) They are connected in parallel.

Problem 2 (Purcell)**.** A conducting rod is pulled to the right at speed v while maintaining contact with two rails. A magnetic field points into the page. We know that an induced emf will cause a current to flow in the counterclockwise direction around the loop. Now, the magnetic force $q\mathbf{u} \times \mathbf{B}$ is perpendicular to the velocity $\mathbf{u}$ of the moving charges, so it can't do work on them. What's going on here? Is the magnetic force doing work or not? If not, then what is? There is definitely something doing work because the wire will heat up.



Problem 3 (Purcell)**.** An infinite solenoid with radius b has n turns per unit length. The current varies in time according to $I(t) = I_0 \cos \omega t$ (with positive defined as shown below). A ring with radius $r < b$ and resistance R is centered on the solenoid's axis, with its plane perpendicular to the axis.

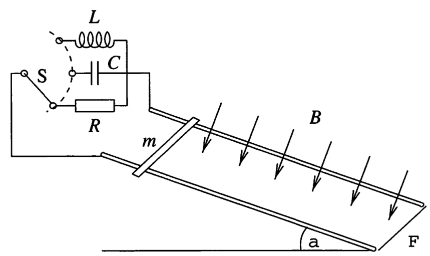


(a) What is the induced current in the ring?

(b) A given little piece of the ring will feel a agnetic force. For what values of t is this force maximum?

(c) What is the effect of the force on the ring? That is, does the force cause the ring to translate, spin, flip over, stretch/shrink, etc.?

Problem 4 (PPP)**.** A homogeneous field of magnetic induction $\mathbf{B}$ is perpendicular to a track of width ℓ which is inclined at an angle α to the horizontal. A frictionless conducting rod of mass m straddles the two rails of the track as shown in the figure below.



How does the rod move, after being released orm rest, if the circuit formed by the rod and thex track is closed by:

(a) a resistor of resistance R,

(b) a capacitor of capacitance C, or

(c) a coil of inductance L?

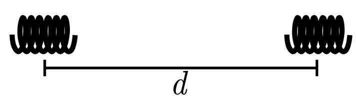
Problem 5. Show that the energy stored in two inductors of inductances L_1, L_2 and mutual inductance M is given by,

$$U(I_1, I_2) = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2.$$

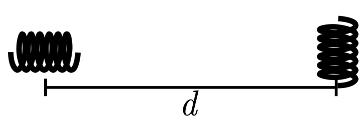
Hint: imagine "charging up" one inductor first and then the next one.

Problem 6. Suppose you have two identical solenoids of radii r, length l, and N turns separated by a distance $d \gg l \gg r$.

(a) Suppose their axes are aligned as shown below. What is their mutual inductance?



(b) Suppose they are now aligned perpendicular to each other as shown below. What is their mutual inductance?



(c) Using the result from problem 5, find the force between the solenoids in each case.

Problem 7. In class, we discussed the falling of a magnet through a metal tube. In this problem, we estimate the terminal speed of the magnet. A magnet of magnetic moment μ and mass m falls through a metal tube of radius R with speed v.

(a) Using physical reasoning and dimensional analysis, estimate the force on the magnet from the tube.

(b) Finish by determining the terminal speed of the magnet.

Problem 8 (EFPhO)**.** We worked with solenoids far away from each other, but now we deal with two that are right next to each other. You have two solenoids, each with N turns, length l, and cross sectional areas $A_1 < A_2 \ll l^2$. They are each connected to a constant current source of magnitude

I and then the smaller solenoid is placed inside the other one such that the centers of the solenoids are a distance $x < l$ apart.

(a) Assuming that $A_1, A_2 \ll x^2, (l-x)^2$, find the total energy E_m of the magnetic fields in this system.

(b) Find the electromotive forces $\mathcal{E}_1$ and $\mathcal{E}_2$ generated on the coils when one is pulled out with a speed v.

(c) Find the force F needed to pull one coil outwards.

Problem 9. A charge of charge q and mass m is orbiting in a circle of radius r due to a uniform magnetic field of magnitude B. If the magnitude of the magnetic field is slowly increased to $2B$ – slow enough such that the charge remains in an approximately circular orbit – what is the final radius of the charge's orbit?

2.2 Challenge Problems

Problem 10. 2023 USAPhO, B1

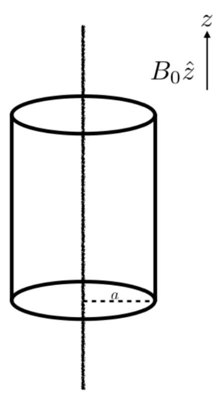
Problem 11. 2009 APhO, T2

Problem 12. 2011 APhO, T1

2.3 USAPhO Practice

Problem 13 (USAPhO 2020 A1)**.** An infinitely long wire with linear charge density $-\lambda$ lies along the z-axis. An infinitely long insulating cylindrical shell of radius a is concentric with the wire and can rotate freely about the z-axis. The shell has moment of inertia per unit length I. Charge is uniformly distributed on the shell, with surface charge density $\frac{\lambda}{2\pi a}$.

The system is immersed in an external magnetic field $B_0\hat{\mathbf{z}}$, and is initially at rest. Starting at $t = 0$, the external magnetic field is slowly reduced to zero over a time $T \gg a/c$, where c is the speed of light.



(a) Find an expression of the final angular velocity ω of the cylinder in terms of the symbols given and other constants.

(b) You may be surprised that the expression you find above is not zero! However, the electric and magnetic fields can have angular momentum. Analogous to the "regular" angular momentum definition, the EM field angular momentum per unit volume at a displacement $\mathbf{r}$ from the axis of rotation is:

$$\mathcal{L}(\mathbf{r}) = \mathbf{r} \times \mathcal{P}(\mathbf{r}).$$

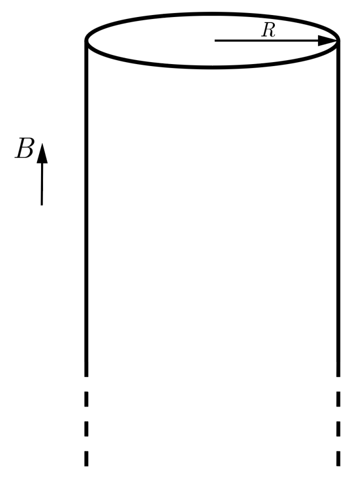
$\mathcal{P}(\mathbf{r})$ is a vector analogous to momentum, given by

$$\mathcal{P}(\mathbf{r}) = \alpha \cdot (\mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})),$$

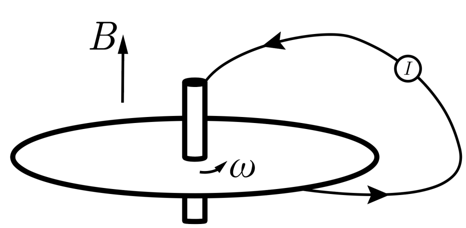
where α is some proportionality constant. Find an expression for α in terms of given variables and fundamental constants.

Problem 14. Magnetic braking is a contactless braking system which uses electromagnetic induction and magnetic interactions to its advantage. We begin by exploring simpler scenarios involving conductors and magnetic fields.

(a) First, suppose we have a conducting cylinder with radius R and length $L \gg R$. It is placed inside a uniform magnetic field with magnitude B and direction parallel to its axis. After a long time, find the charge distribution on the conducting cylinder.



(b) Now, we replace the cylinder with a disk of radius R and attach a current source of magnitude I to the center and outer radius of the disk as shown below. What is the torque that is applied to the disk due to the magnetic field?



(c) What is the EMF applied by the current source as a function of time in order to maintain the constant current I?

(d) In magnetic braking, the magnetic field is only applied to a small portion of the disk. Suppose that it is applied on some sector of size θ as shown below. Furthermore, assume for ease that the disk is actually produced of many radial wires as shown below. What is the braking torque experienced by the disk? Also assume for ease that each "triangle" in the disk can only flow current within its own loop.